MATH 2010 Advanced Calculus I
Suggested Solutions for Homework 3

10.2, Q18 Find the slope of the curve$x = (t), y = g(t)$ at the given value of t.

$$x \sin t + 2x = t, \qquad t \sin t - 2t = y, \qquad t = \pi.$$

Solution:

Differentiating with respect to t yields

$$\frac{dx}{dt} \sin t + x \cos t + 2\frac{dx}{dt} = 1, \qquad \sin t + t \cos t - 2 = \frac{dy}{dt}$$

Then

$$\frac{dx}{dt} = \frac{1 - x\cos t}{2 + \sin t}$$

and

$$\frac{dy}{dx} = \frac{\sin t + t\cos t - 2}{\frac{1-x\cos t}{2+\sin t}}.$$

When $t = \pi$, we have $x = \frac{\pi}{2}$ and

$$\frac{dy}{dx}\Big|_{t=\pi} = \frac{-4\pi - 8}{2 + \pi}.$$

10.2, Q28 Find the length of the curve

$$x = \frac{(2t+3)^{\frac{3}{2}}}{3}, \qquad y = t + \frac{t^2}{2}, \qquad 0 \le t \le 3.$$

Solution:

Note that

$$\frac{dx}{dt} = (2t+3)^{\frac{1}{2}}, \qquad \frac{dy}{dt} = 1 + t$$

Since $0 \le t \le 3$, then, we have

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{2t + 3 + 1 + t^2 + 2t} = |t + 2|$$

and the length is

$$\int_0^3 (t+2)dt = (\frac{t^2}{2} + 2t)\Big|_0^3 = \frac{21}{2}.$$

10.2, Q30 Find the length of the curve

$$x = \ln(\sec t + \tan t) - \sin t, \qquad y = \cos t, \qquad 0 \le t \le \pi/3.$$

Solution:

Since

$$\frac{dx}{dt} = \frac{\tan t \sec t + \sec^2 t}{\sec t + \tan t} - \cos t = \sec t - \cos t, \qquad \frac{dy}{dt} = -\sin t.$$

Then

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t}.$$

Note that $\sqrt{\tan^2 t} = |\tan t| = \tan t$ for $0 \le t \le \pi/3$, the length is

$$\int_0^{\pi/3} \tan t \, dt = -\ln \cos t \Big|_0^{\pi/3} = \ln 2.$$

Replace the polar equations in Q36, Q42, Q52 with equivalent Cartesian equations. Then describe or identify the graph.

10.3, Q36

$$r^2 = 4r \sin\theta.$$

Solution: Since

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

then we write $r^2 = 4r\sin\theta$ as

$$x^2 + y^2 = 4y$$

which is equivalent to

$$x^2 + (y-2)^2 = 4.$$

The graph is the circle centered at (0, 2) with radius 2.

10.3, Q42

$$r\sin\theta = \ln r + \ln\cos\theta.$$

Solution:

Note that $\ln r + \ln\cos\theta = \ln(r\cos\theta) = \ln x$, the we have

$$y = \ln x$$

This is the graph of the natural exponential function.

10.3, Q52

$$r\sin(\frac{2\pi}{3} - \theta) = 5.$$

Solution:

We write

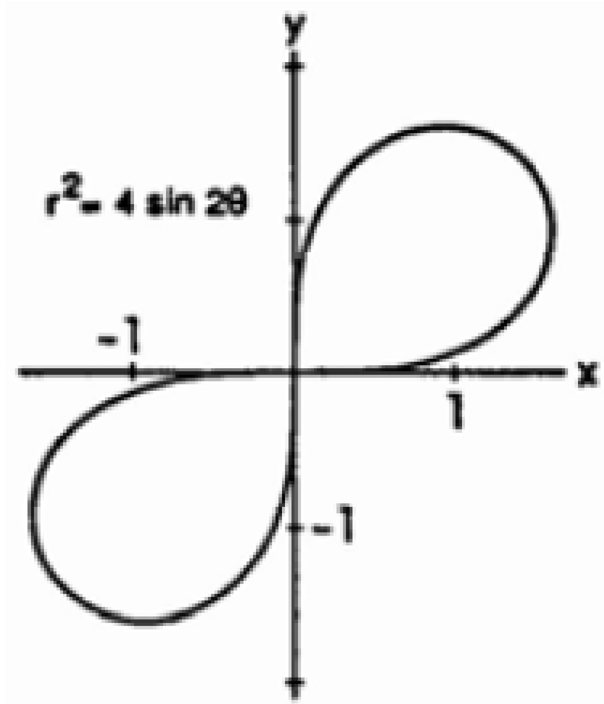
$$r\sin(\frac{2\pi}{3} - \theta) = r\sin\frac{2\pi}{3}\cos\theta - r\cos\frac{2\pi}{3}\sin\theta = \frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = \frac{\sqrt{3}}{2}x + \frac{1}{2}y.$$

It follows that

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5 \Rightarrow \sqrt{3}x + y = 10$$

The graph is the line with slope $-\sqrt{3}$ and intercept with y-axis at (0, 10).

Graph



10.4, Q14 Graph the lemniscates for

$$r^2 = 4 \sin \theta.$$

What symmetries do these curves have?

Solution:

Since (r, θ) on the graph $\Rightarrow$ $(-r, \theta)$ is on the graph. Then the graph is symmetric about the origin. But $4 \sin 2(-\theta) \neq r^2$ and

$$4 \sin 2(\pi - \theta) = 4 \sin(2\pi - 2\theta) = 4 \sin(-2\theta) = -4 \sin(2\theta) \neq r^2$$

It follows that the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis.

Find the curve's unit tangent vector in Q4 and Q6. Also, find the length of the indicated portion of the curve.

12.3, Q4

$$r(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \qquad 0 \leq t \leq 3.$$

Solution:

The velocity vector $\mathbf{v} = \frac{dr}{dt} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $|\mathbf{v}| = \sqrt{3}$. Then the unit tangent vector is

$$\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

and the length is

$$\int_0^3 \sqrt{3}\, dt = 3\sqrt{3}.$$

12.3, Q6

$$r(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \qquad 1 \le t \le 2.$$

Solution:

The velocity vector $\mathbf{v} = \frac{dr}{dt} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k}$ and $|\mathbf{v}| = \sqrt{441t^4} = 21t^2$. Then the unit tangent vector is

$$\mathbf{v} = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$$

and the length is

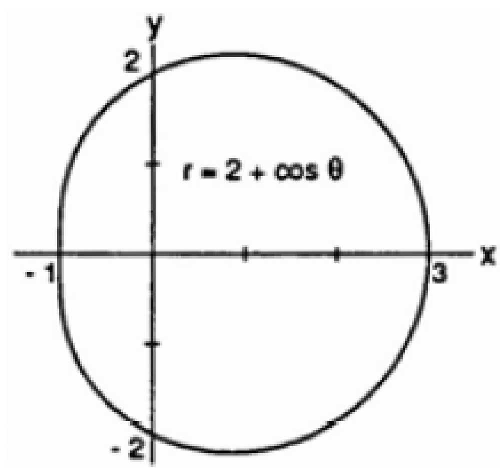
$$\int_1^2 21t^2\, dt = 49.$$

10.4, Q28 Oval limaçons

$$\text{(a)}.\, r = 2 + \cos\theta, \qquad \text{(b)}.\, r = -2 + \sin\theta$$

Graph

(a)



Graph

(b)